OCR
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AS Level Further Mathematics B (MEI)
Y410 Core Pure
Sample Question Paper

Date – Morning/Afternoon
Time allowed: 1 hour 15 minutes

OCR supplied materials:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)
- Scientific or graphical calculator

MODEL SOLUTIONS

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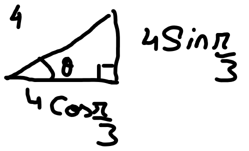
INSTRUCTIONS
- Use black ink. HB pencil may be used for graphs and diagrams only.
- Complete the boxes provided on the Printed Answer Booklet with your name, centre number and candidate number.
- Answer all the questions.
- Write your answer to each question in the space provided in the Printed Answer Booklet.
- Additional paper may be used if necessary but you must clearly show your candidate number, centre number and question number(s).
- Do not write in the bar codes.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION
- The total number of marks for this paper is 60.
- The marks for each question or part question are shown in brackets [ ].
- You are advised that an answer may receive no marks unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of 12 pages. The Question Paper consists of 4 pages.

Answer all the questions.

1 The complex number z_1 is $1+i$ and the complex number z_2 has modulus 4 and argument $\frac{\pi}{3}$.

(i) Express z_2 in the form $a+bi$, giving a and b in exact form. [2]



$\theta = \frac{\pi}{3}$

$$z_2 = 4\cos\frac{\pi}{3} + \left(4\sin\frac{\pi}{3}\right)i$$

$$= \underline{2 + 2\sqrt{3}\,i}$$

(ii) Express $\frac{z_2}{z_1}$ in the form $c+di$, giving c and d in exact form. [2]

$$\frac{z_2}{z_1} = \frac{2+2\sqrt{3}\,i}{1+i}$$

$$= \frac{(2+2\sqrt{3}\,i)}{1+i} \times \frac{(1-i)}{(1-i)}$$

$$= \frac{2+2\sqrt{3}i - 2i - 2\sqrt{3}i^2}{1-i^2}$$

$$= \frac{2+2\sqrt{3} + (2\sqrt{3}-2)i}{2}$$

$$= (1+\sqrt{3}) + (\sqrt{3}-1)i$$

2 (i) Describe fully the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. [2]

Shear with x-axis fixed and $(0, 1)$ transforms to $(2, 1)$.

(ii) A triangle of area 5 square units undergoes the transformation represented by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Explaining your reasoning, find the area of the image of the triangle following this transformation. [2]

Shear transformation preserves the area. $\therefore$ New area = 5.

3 (i) Write down, in complex form, the equation of the locus represented by the circle in the Argand diagram shown in Fig. 3. [2]

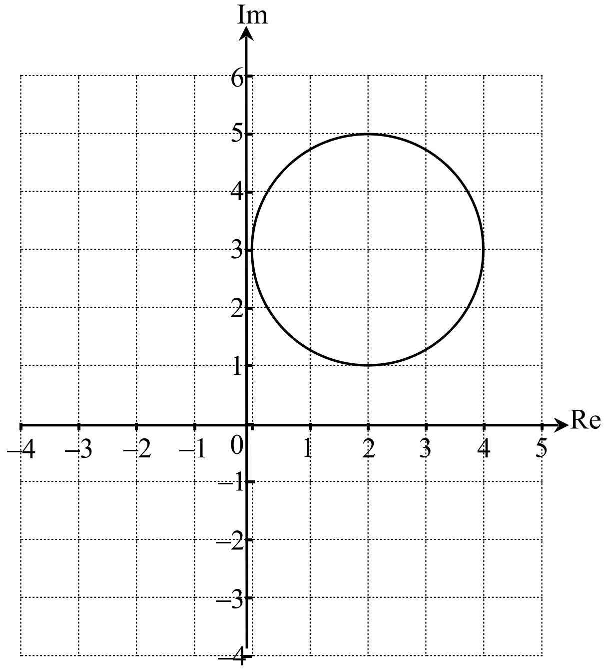


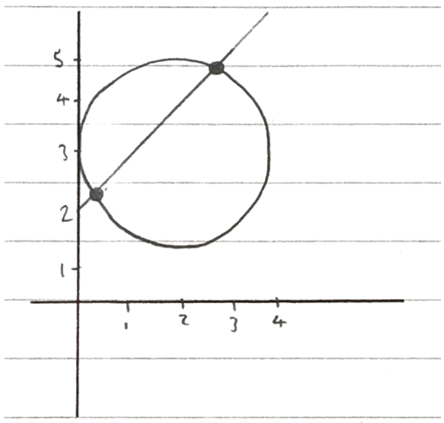
Fig. 3

Centre = $(2, 3)$

Radius = 2

Equation: $|z - (2+3i)| = 2$

(ii) On the copy of Fig. 3 in the Printed Answer Booklet mark with a cross any point(s) on the circle for which $\arg(z-2i)=\frac{\pi}{4}$. [2]



4 (i) Find the coordinates of the point where the following three planes intersect. Give your answers in terms of a.

$$\begin{aligned} x-2y-\ z&=\ 6\\ 3x+\ y+5z&=-4\\ -4x+2y-3z&=\ a \end{aligned}$$

[4]

$$\Rightarrow \begin{pmatrix} 1 & -2 & -1 \\ 3 & 1 & 5 \\ -4 & 2 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -2 & -1 \\ 3 & 1 & 5 \\ -4 & 2 & -3 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ -4 \\ a \end{pmatrix}$$

$$\det = (-3-10)+2(-9+20)-(6+4)$$

$$= -1$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-1}{1}\begin{pmatrix} -3-10 & -(6+2) & -10+1 \\ -(-9+20) & -3-4 & -(5+3) \\ 6+4 & -(2-8) & 1+6 \end{pmatrix}\begin{pmatrix} 6 \\ -4 \\ a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 13 & 8 & 9 \\ 11 & 7 & 8 \\ -10 & -6 & -7 \end{pmatrix} \begin{pmatrix} 6 \\ -4 \\ a \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 46+9a \\ 38+8a \\ -36-7a \end{pmatrix}$$

$\therefore$ Co-ordinates are $(x, y, z) =$

$$(46+9a,\ 38+8a,\ -36-7a)$$

(ii) Determine whether the intersection of the three planes could be on the z-axis. [2]

If they intersect on the z axis then $x = 0$ and $y = 0$ would need to both satisfy the co-ordinates in the previous part.

$\therefore$ $46 + 9a = 0$

$9a = -46$

$a = -46/9$

Subs. in y: $38 + 8\left(-\frac{46}{9}\right) = -\frac{26}{9} \neq 0$

$\therefore$ x and y both can't be 0. Hence, the planes don't intersect on the z axis.

5 The cubic equation $x^3 - 4x^2 + px + q = 0$ has roots $\alpha, \frac{2}{\alpha}$ and $\alpha + \frac{2}{\alpha}$.

Find

- the values of the roots of the equation,
- the value of p. [7]

$$x^3 - 4x^2 + px + q = 0$$

$$\alpha + \frac{2}{\alpha} + \alpha + \frac{2}{\alpha} = 4 \quad \text{as } \Sigma\alpha = -\frac{b}{a}$$

$$\Rightarrow 2\alpha + \frac{4}{\alpha} = 4$$

$$\Rightarrow 2\alpha^2 + 4 = 4\alpha$$

$$\Rightarrow \alpha^2 - 2\alpha + 2 = 0$$

$$\Rightarrow (\alpha - 1)^2 - 1 + 2 = 0$$

$$\Rightarrow (\alpha - 1)^2 + 1 = 0 \Rightarrow \alpha = \underline{\underline{1 \pm i}}$$

Roots: $\alpha, \frac{2}{\alpha}, \alpha + \frac{2}{\alpha}$

$$= 1+i, \frac{2}{1+i}, 1+i+\frac{2}{1+i}$$

$$= 1+i, \frac{2}{1+i} \times \frac{1-i}{1-i}, 1+i+\frac{2}{1+i} \times \frac{1-i}{1-i}$$

$$= 1+i, 1-i, 1+i+1-i$$

$$= 1+i, 1-i, 2$$

Using $1-i$ instead of $1+i$ would give same roots. Because it is a cubic there must be 3 roots.

$$\Sigma \alpha\beta = -\frac{p}{a}$$

$$\Rightarrow p = (1+i)(1-i) + 2(1+i) + 2(1-i)$$

$$= 1 - \cancel{i} + \cancel{i} + 1 + 2 + \cancel{2i} + 2 - \cancel{2i}$$

$$= \underline{\underline{4}}$$

$$\Rightarrow \underline{\underline{p = 4}}$$

6 (i) Show that, when $n = 5$, $\sum_{r=n+1}^{2n} r^2 = 330$. [1]

$$\sum_{r=6}^{10} r^2 = 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = \underline{\underline{330}}$$

(ii) Find, in terms of n, a fully factorised expression for $\sum_{r=n+1}^{2n} r^2$. [4]

$$\sum_{r=n+1}^{2n} r^2 = \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n} r^2$$

$$= \frac{1}{6}(2n)(2n+1)(4n+1) - \frac{1}{6}(n)(n+1)(2n+1)$$

$$= \frac{1}{6}n(2n+1)\left[2(4n+1) - (n+1)\right]$$

$$= \frac{1}{6}n(2n+1)(7n+1)$$

7 The plane Π has equation $3x - 5y + z = 9$.

(i) Show that Π contains

- the point $(4,1,2)$

and

- the vector $\begin{pmatrix}1\\1\\2\end{pmatrix}$. [4]

$3x - 5y + z = 9$

Sub in $(4,1,2)$: $3(4) - 5(1) + 2$

$= 9$ ✓

$\begin{pmatrix}3\\-5\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} = 3 - 5 + 2 = 0$

$\therefore$ The vector $\begin{pmatrix}1\\1\\2\end{pmatrix}$ is perpendicular to the normal vector of the plane. Therefore, $\begin{pmatrix}1\\1\\2\end{pmatrix}$ lies on the plane.

(ii) Determine the equation of a plane which is perpendicular to Π and which passes through $(4,1,2)$. [3]

$\begin{pmatrix}1\\1\\2\end{pmatrix}$ is parallel to Π so it can be used as normal for the new plane

New plane:

$x + y + 2z = d$

$4 + 1 + 2(2) = 9$

$\Rightarrow$ $\underline{\underline{x + y + 2z = 9}}$ is equation of plane

8 In this question you must show detailed reasoning.

(i) Explain why all cubic equations with real coefficients have at least one real root. [2]

If complex number is one of roots, so is its complex conjugate. This means that complex roots occur in pairs. So if an equation has 3 roots and 2 of those are a complex pair, then one root must be real.

(ii) Points representing the three roots of the equation $z^3 + 9z^2 + 27z + 35 = 0$ are plotted on an Argand diagram.

Find the exact area of the triangle which has these three points as its vertices. [7]

$f(-5) = 0 \Rightarrow (z+5)$ is a factor [Factor thm]

$$(z+5)\overline{\big)\, z^3 + 9z^2 + 27z + 35}\quad z^2 + 4z + 7$$

$$z^3 + 5z^2$$

$$4z^2 + 27z + 35$$

$$4z^2 + 20z$$

$$7z + 35$$

$$7z + 35$$

$$\times$$

$\therefore f(z) = (z+5)(z^2 + 4z + 7)$

let $(z^2 + 4z + 7) = 0$

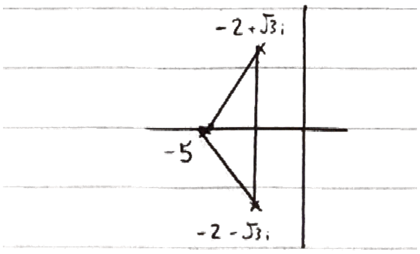
$\Rightarrow (z+2)^2 - 4 + 7 = 0$

$\Rightarrow (z+2)^2 + 3 = 0$

$\Rightarrow z + 2 = \pm\sqrt{3}\, i$

$\Rightarrow z = -2 \pm \sqrt{3}i$

$\therefore$ Roots are -5, $-2+\sqrt{3}i$, $-2-\sqrt{3}i$



Area $= \frac{1}{2} \times 2\sqrt{3} \times (5-2)$

$= \underline{\underline{3\sqrt{3} \text{ sq. units.}}}$

9 You are given that matrix $M = \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$.

(i) Prove that, for all positive integers n, $M^n = \begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}$. [6]

Proof by Induction:

When $n = 1$, $M^1 = \begin{pmatrix} 1-4 & 8 \\ -2 & 1+4 \end{pmatrix}$

$= \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix} = M$

$\therefore$ True for $n = 1$

Assuming true for $n = k$, i.e.

$M^k = \begin{pmatrix} 1-4k & 8k \\ -2k & 1+4k \end{pmatrix}$

Checking for $n = k+1$

$$M^{k+1} = M^k \times M = \begin{pmatrix} 1-4k & 8k \\ -2k & 1+4k \end{pmatrix}\begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} -3-4k & 8+8k \\ -2-2k & 5+4k \end{pmatrix}$$

$$= \begin{pmatrix} 1-4(k+1) & 8(1+k) \\ -2(k+1) & 1+4(k+1) \end{pmatrix}$$

As needed. ✓

$\therefore$ If true for $n=k$, it is true for $n=k+1$ and because it's true for $n=1$, it is true for $n \in \mathbb{Z}^+$ by mathematical induction.

(ii) Determine the equation of the line of invariant points of the transformation represented by the matrix M. [3]

$$\begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$-3x+8y = x$ $\qquad$ $-2x+5y = y$

$8y = 4x$ $\qquad$ $4y = 2x$

$2y = x$ $\qquad$ $2y = x$

$$\therefore \; y = \tfrac{1}{2}x$$

It is claimed that the answer to part (ii) is also a line of invariant points of the transformation represented by the matrix M^n, for any positive integer n.

(iii) Explain geometrically why this claim is true. [2]

M^n is transforming n times. Line is unchanged under M so each time transformation is done, line is unaffected.

(iv) Verify algebraically that this claim is true. [3]

$$\begin{pmatrix} 1-4n & 8n \\ -2n & 1+4n \end{pmatrix}\begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix} = \begin{pmatrix} x - 4xn + 4xn \\ -2nx + \frac{1}{2}x + 2nx \end{pmatrix}$$

$$= \begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix}$$

$\therefore$ It is invariant for all n.

END OF QUESTION PAPER